

# Completeness of infinite-dimensional Lie groups in their left uniformity

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## Abstract

We prove completeness for the main examples of infinite-dimensional Lie groups and some related topological groups. Consider a sequence  $G_1 \subseteq G_2 \subseteq \dots$  of topological groups  $G_n$  such that  $G_n$  is a subgroup of  $G_{n+1}$  and the latter induces the given topology on  $G_n$ , for each  $n \in \mathbb{N}$ . Let  $G$  be the direct limit of the sequence in the category of topological groups. We show that  $G$  induces the given topology on each  $G_n$  whenever  $\bigcup_{n \in \mathbb{N}} V_1 V_2 \dots V_n$  is an identity neighbourhood in  $G$  for all identity neighbourhoods  $V_n \subseteq G_n$ . If, moreover, each  $G_n$  is complete, then  $G$  is complete. We also show that the weak direct product  $\bigoplus_{j \in J} G_j$  is complete for each family  $(G_j)_{j \in J}$  of complete Lie groups  $G_j$ . As a consequence, every strict direct limit  $G = \bigcup_{n \in \mathbb{N}} G_n$  of finite-dimensional Lie groups is complete, as well as the diffeomorphism group  $\text{Diff}_c(M)$  of a paracompact finite-dimensional smooth manifold  $M$  and the test function group  $C_c^k(M, H)$ , for each  $k \in \mathbb{N}_0 \cup \{\infty\}$  and complete Lie group  $H$  modelled on a complete locally convex space.

**Classification:** 22E65 (primary); 22A05, 22E67, 46A13, 46M40, 58D05

**Key words:** infinite-dimensional Lie group; locally convex group; weak direct product; box product; direct sum; (LB)-space; inductive limit; direct limit; ascending union; strict direct sequence; ascending sequence; direct system; product set; strong topology; bamboo shoot topology; compact support; test function group; diffeomorphism group; Banach-Lie group; left uniform structure; one-sided uniformity; Cauchy net; Cauchy filter; Cauchy sequence; strong (ILB)-Lie group; projective limit; inverse limit

## Introduction and statement of the main results

Our main goal is to study completeness for Lie groups modelled on locally convex spaces (in the sense of [20], [30], [42], cf. also [31], [39], [40]), and more generally completeness of topological groups, as far as this is useful for the

main goal. Here completeness refers to the left uniform structure on  $G$  (see, e.g., [32] for the latter). The topological groups under consideration need not be Hausdorff (unless we say so explicitly). It is well-known that every Lie group  $G$  modelled on a Banach space  $E$  is complete (see Proposition 1 in [12, Chapter III, §1.1]), as the left uniform structure and the one induced by the additive group of the Banach space coincide on some identity neighbourhood  $U \subseteq G$  which is homeomorphic to a closed 0-neighbourhood  $V \subseteq E$ .

Projective limits of complete Hausdorff groups being complete, this implies that many Fréchet-Lie groups<sup>1</sup> are complete, e.g. the mapping groups

$$C^\infty(M, H) = \varprojlim C^k(M, H)$$

which are the projective limit of the Banach-Lie groups  $C^k(M, H)$  for  $k \in \mathbb{N}_0$ , for each compact smooth manifold  $M$  and Banach-Lie group  $H$ . Of course, completeness properties of locally convex spaces  $E$  (which furnish examples of abelian Lie groups  $(E, +)$ ) are a standard topic in functional analysis. Moreover, completeness properties of topological groups have been studied (see [47] and the references therein, also [2]). However, no systematic study of completeness properties of infinite-dimensional Lie groups is available so far. For example, it is an open question whether completeness of the modelling space implies completeness for a Lie group (see Problem II.9 in the survey [42]). The current article strives to develop specific tools which enable completeness to be shown for important classes of infinite-dimensional Lie groups, under natural hypotheses. Our main results, Theorems A and B, are devoted to completeness properties of direct limits. Recall that an ascending sequence

$$G_1 \subseteq G_2 \subseteq \cdots$$

of topological groups  $(G_n, \mathcal{O}_n)$  is a *direct sequence of topological groups* if, for each  $n \in \mathbb{N}$ , the inclusion map  $j_{n+1,n}: (G_n, \mathcal{O}_n) \rightarrow (G_{n+1}, \mathcal{O}_{n+1})$  is a continuous homomorphism. If, moreover, each  $j_{n+1,n}$  is a topological embedding (i.e., a homeomorphism onto its image), then the direct sequence is called *strict*. Give  $G := \bigcup_{n \in \mathbb{N}} G_n$  the unique group structure for which each inclusion map  $j_n: G_n \rightarrow G$  is a group homomorphism. There is a finest group topology  $\mathcal{O}_{TG}$  on  $G$  making each  $j_n$  continuous;  $(G, \mathcal{O}_{TG})$  is called the *direct limit topological group*.<sup>2</sup> The topology  $\mathcal{O}_{TG}$  must not be confused with the

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<sup>1</sup>As usual, Lie groups modelled on Fréchet spaces (resp., Banach spaces) are called Fréchet-Lie groups (resp., Banach-Lie groups) in the following.

<sup>2</sup>A topology on a group  $G$  is called a *group topology* if it makes  $G$  a topological group.

final topology  $\mathcal{O}_{DL}$  on  $G$  with respect to the inclusion maps  $j_n$ , which makes  $(G, \mathcal{O}_{DL})$  the direct limit topological space. It is clear that  $\mathcal{O}_{TG} \subseteq \mathcal{O}_{DL}$ , but examples show that equality need not hold (see, e.g., [49] and [54]).

Let  $\mathcal{O}$  be a group topology on  $G$  making each  $j_n$  continuous. Following [26], we say that *product sets are large* in  $(G, \mathcal{O})$  if<sup>3</sup>

$$\bigcup_{n \in \mathbb{N}} V_1 V_2 \cdots V_n \quad (1)$$

is an identity neighbourhood in  $(G, \mathcal{O})$  for all identity neighbourhoods  $V_n$  in  $(G_n, \mathcal{O}_n)$ . If product sets are large in  $(G, \mathcal{O})$ , then  $\mathcal{O} = \mathcal{O}_{TG}$  (see [26, Proposition 11.8]), and moreover the product sets as in (1) form a basis of identity neighbourhoods for  $(G, \mathcal{O})$  (as we recall in Lemma 3.2). We mention that sets of the form

$$\bigcup_{n \in \mathbb{N}} (V_n \cdots V_2 V_1)(V_1 V_2 \cdots V_n)$$

(so-called “bamboo shoots”) were already used in [49] and [33] to obtain tangible descriptions of the topology  $\mathcal{O}_{TG}$  in well-behaved situations. Our main result can be formulated as follows:

**Theorem A** *Let  $G_1 \subseteq G_2 \subseteq \cdots$  be a strict direct sequence of topological groups and  $G := \bigcup_{n \in \mathbb{N}} G_n$  be its direct limit topological group.*

- (a) *If product sets are large in  $G$ , then each inclusion map  $G_n \rightarrow G$  is a topological embedding.*
- (b) *If product sets are large in  $G$  and each  $G_n$  is complete, then also  $G$  is complete.*

We mention that  $(G, \mathcal{O}_{TG})$  is Hausdorff if each  $G_n$  is Hausdorff and the inclusion map  $G_n \rightarrow G$  a topological embedding.<sup>4</sup>

Theorem A and its proof were inspired by Bourbaki’s discussion of completeness for strict direct limits of complete locally convex spaces [11].

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<sup>3</sup>In [3],  $(G, \mathcal{O})$  is then said to *carry the strong topology*.

<sup>4</sup>If  $e \neq x \in G_n$ , there is an open identity neighbourhood  $V \subseteq G_n$  with  $x \notin V$ . Let  $W \subseteq G$  be an open identity neighbourhood such that  $W \cap G_n = V$ . Then  $x \notin W$ . Hence  $\overline{\{e\}} = \{e\}$  in  $G$  and thus  $G$  is Hausdorff.

Now consider a family  $(G_j)_{j \in J}$  of Lie groups  $G_j$  modelled on locally convex spaces  $E_j$ . Then the so-called *weak direct product*

$$G := \bigoplus_{j \in J} G_j := \left\{ (x_j)_{j \in J} \in \prod_{j \in J} G_j : x_j = e \text{ for almost all } j \right\}$$

can be made a Lie group modelled on the locally convex direct sum  $E := \bigoplus_{j \in J} E_j$  in such a way that for some  $C^\infty$ -diffeomorphisms  $\phi_j : U_j \rightarrow V_j$  from an open identity neighbourhood  $U_j \subseteq G_j$  onto an open 0-neighbourhood  $V_j \subseteq E_j$  with  $\phi_j(e) = 0$ , the set

$$\bigoplus_{j \in J} U_j := G \cap \prod_{j \in J} U_j$$

is an open identity neighbourhood in  $G$  and the map

$$\bigoplus_{j \in J} \phi_j : \bigoplus_{j \in J} U_j \rightarrow \bigoplus_{j \in J} V_j \subseteq E, \quad (x_j)_{j \in J} \mapsto (\phi_j(x_j))_{j \in J}$$

is a  $C^\infty$ -diffeomorphism (see [22]). If  $J$  is countable, then the topological group underlying the weak direct product  $\bigoplus_{j \in J} G_j$  is the small box product of the topological groups  $G_j$  (as in [4]). If  $J$  is uncountable, then the weak direct product and the small box product still coincide as groups, but the box topology is coarser and can be properly coarser. For example, this happens for the family  $(\mathbb{R})_{j \in J}$  for an uncountable set  $J$ . The weak direct product

$$\mathbb{R}^{(J)} := \bigoplus_{j \in J} \mathbb{R}$$

then coincides with the locally convex direct sum, whose topology differs from the box topology,<sup>5</sup> as is well known (see, e.g., [50]). We show:

**Theorem B.** *Let  $(G_j)_{j \in J}$  be a family of Lie groups  $G_j$  modelled on locally convex spaces. If each  $G_j$  is complete (resp., sequentially complete), then also the weak direct product  $\bigoplus_{j \in J} G_j$  is complete (resp., sequentially complete).*

Similarly, one finds that the small box product of each family of complete

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<sup>5</sup>In fact,  $\{(x_j)_{j \in J} \in \mathbb{R}^{(J)} : \sum_{j \in J} |x_j| < 1\}$  is a 0-neighbourhood in the locally convex direct sum which cannot contain any box  $\bigoplus_{j \in J} ]-q_j, q_j[$  with  $q_j \in ]0, \infty[ \cap \mathbb{Q} =: C$  as one of the sets  $J_q := \{j \in J : q_j = q\}$  with  $q \in C$  must be uncountable and hence infinite.

(resp., sequentially complete) topological groups is complete (resp., sequentially complete), see Example 4.2.

We now explain how the main results (and further findings) can be used to establish completeness for infinite-dimensional Lie groups within the main classes of examples (as listed in [42, pp. 3-4]), and related topological groups.

**Direct limits of finite-dimensional Lie groups.** If  $G_1 \subseteq G_2 \subseteq \dots$  is a direct sequence of topological groups and the direct limit topology  $\mathcal{O}_{DL}$  on  $G = \bigcup_{n \in \mathbb{N}} G_n$  makes  $G$  a topological group (i.e., if  $\mathcal{O}_{TG} = \mathcal{O}_{DL}$ ), then product sets are large in  $(G, \mathcal{O}_{TG})$  (see [21, Proposition 11.3]). Thus Theorem A entails:

*If  $\mathcal{O}_{TG} = \mathcal{O}_{DL}$  on  $G = \bigcup_{n \in \mathbb{N}} G_n$  for a strict direct sequence  $G_1 \subseteq G_2 \subseteq \dots$  of complete topological groups, then  $(G, \mathcal{O}_{TG})$  is complete.*

We mention that  $\mathcal{O}_{TG} = \mathcal{O}_{DL}$  on  $G = \bigcup_{n \in \mathbb{N}} G_n$  for each direct sequence  $G_1 \subseteq G_2 \subseteq \dots$  of locally compact Hausdorff topological groups (see [49] and [33]). Hence every strict direct limit  $G = \bigcup_{n \in \mathbb{N}} G_n$  of locally compact Hausdorff topological groups  $G_1 \subseteq G_2 \subseteq \dots$  is complete. In particular, the Lie groups  $\varinjlim G_n$  (as in [24]) are complete for each strict direct sequence  $G_1 \subseteq G_2 \subseteq \dots$  of finite-dimensional Lie groups.<sup>6</sup>

As shown in [29],  $\mathcal{O}_{TG} = \mathcal{O}_{DL}$  also holds for  $G = \bigcup_{n \in \mathbb{N}} G_n$  if each topological group  $G_n$  is a  $k_\omega$ -space (i.e., a direct limit of a direct sequence of compact Hausdorff topological spaces) or *locally  $k_\omega$*  (viz. a Hausdorff space in which every point has an open neighbourhood which is a  $k_\omega$ -space).<sup>7</sup> It is known that abelian topological groups which are  $k_\omega$ -spaces are complete [46]. Since direct limits  $\bigcup_{n \in \mathbb{N}} G_n$  of locally compact Hausdorff groups are locally  $k_\omega$  and have an open subgroup which is  $k_\omega$  (see [29]), this furnishes an alternative proof for the completeness of direct limits  $\bigcup_{n \in \mathbb{N}} G_n$  of locally compact Hausdorff *abelian* groups (even irrespective of strictness of the direct sequence).

**Diffeomorphism groups.** For  $M$  a paracompact finite-dimensional smooth manifold, consider the group  $\text{Diff}_c(M)$  of all  $C^\infty$ -diffeomorphisms  $\phi: M \rightarrow M$  with compact support (in the sense that  $\phi(x) = x$  for  $x$  outside some compact set). Then  $\text{Diff}_c(M)$  is a Lie group modelled on the space of smooth

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<sup>6</sup>The Lie group structure on important examples of such groups (like  $\text{GL}_\infty(\mathbb{R}) = \varinjlim \text{GL}_n(\mathbb{R})$  and  $\text{SL}_\infty(\mathbb{R}) = \varinjlim \text{SL}_n(\mathbb{R})$ ) was already constructed in [38]; cf. also [41].

<sup>7</sup>See [19] and [29] for further information on  $k_\omega$ -spaces, and the references therein.

compactly supported vector fields on  $M$  (and  $\text{Diff}(M)$  can be made a Lie group with  $\text{Diff}_c(M)$  as an open normal subgroup), see [39] (or also [27] and [30] if  $M$  is  $\sigma$ -compact). For each compact subset  $K \subseteq M$ ,

$$\text{Diff}_K(M) := \{\phi \in \text{Diff}_c(M) : (\forall x \in M \setminus K) \phi(x) = x\}$$

is a Lie subgroup of  $\text{Diff}_c(M)$ , modelled on the Fréchet space of all smooth vector fields supported in  $K$ . If  $M$  is  $\sigma$ -compact and  $K_1 \subseteq K_2 \subseteq \dots$  an exhaustion of  $M$  by compact sets,<sup>8</sup> then

$$\text{Diff}_{K_1}(M) \subseteq \text{Diff}_{K_2}(M) \subseteq \dots \quad (2)$$

is a strict direct sequence of Lie groups. By [26, Example 11.7], the product map

$$\pi : \bigoplus_{n \in \mathbb{N}} \text{Diff}_{K_n}(M) \rightarrow \text{Diff}_c(M)$$

taking  $(\phi_1, \dots, \phi_n, \text{id}_M, \text{id}_M, \dots)$  to  $\phi_1 \circ \dots \circ \phi_n$  admits a smooth local section around  $\text{id}_M$  (in the spirit of fragmentation techniques familiar in the theory of diffeomorphism groups, cf. [7]). By [26, Remark 11.5 and Proposition 11.8], this implies that product sets are large in  $\text{Diff}_c(M)$  and  $\text{Diff}_c(M)$  is the direct limit topological group of (2), as recorded in [26, Proposition 5.4] (see also Remark 1 and Proposition 1 in [6] for these arguments).<sup>9</sup>

Now  $\text{Diff}_{K_n}(M)$  is a strong (ILB)-Lie group (as considered in [44]) for each  $n \in \mathbb{N}$ . Using that strong (ILB)-Lie groups are complete (see Proposition 5.1 and Remark 5.2 (a)), Theorem A implies that  $\text{Diff}_c(M)$  is complete for  $\sigma$ -compact  $M$  (see Remark 5.2 (b) and (c) for details).

If  $M$  is merely paracompact and  $(M_j)_{j \in J}$  its family of connected components, then  $\text{Diff}_c(M)$  has an open subgroup  $G$  which is isomorphic to the weak direct product

$$\bigoplus_{j \in J} \text{Diff}_c(M_j)$$

as a Lie group, and we deduce with Theorem B that  $G$  (and hence  $\text{Diff}_c(M)$ ) is complete also for paracompact  $M$ .

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<sup>8</sup>Thus  $M = \bigcup_{n \in \mathbb{N}} K_n$  and  $K_n \subseteq K_{n+1}^0$  for each  $n \in \mathbb{N}$ .

<sup>9</sup>We mention that the topology on the Lie group  $\text{Diff}_c(M)$  coincides with the Whitney  $C^\infty$ -topology used in [6]; this is clear from the description of this topology in [36] (see also [35] for a detailed account).

The completeness of diffeomorphism groups contrasts the incompleteness of many groups of homeomorphisms, among which one finds typical examples of metrizable topological groups which cannot be completed as the sets of Cauchy sequences for the left and right uniformity do not coincide (see [18]).

**Mapping groups and gauge groups.** Among the prime examples of infinite-dimensional Lie groups are the Lie groups  $C^k(M, H)$  of  $C^k$ -maps from a compact manifold  $M$  to a Lie group  $H$  for  $k \in \mathbb{N}_0 \cup \{\infty\}$  (notably the *loop groups* with  $M = \mathbb{S}^1$  the circle group [45]), see [40] and [44]. More generally, if  $M$  is a paracompact finite-dimensional smooth manifold and  $H$  a Lie group modelled on a locally convex space  $E$ , then there is a natural Lie group structure on the group  $C_c^k(M, H)$  of all  $C^k$ -maps  $\gamma: M \rightarrow H$  whose support

$$\text{supp}(\gamma) := \overline{\{x \in M : \gamma(x) \neq e\}}$$

is compact (where  $e$  is the neutral element of  $H$ ), which is modelled on the locally convex space  $C_c^k(M, E)$ ; see [21] (also [1]) if  $M$  is  $\sigma$ -compact; in the general case, we let  $(M_j)_{j \in J}$  be the family of connected components of  $M$  and use the group isomorphism

$$C_c^k(M, H) \rightarrow \bigoplus_{j \in J} C_c^k(M_j, H), \quad \gamma \mapsto (\gamma|_{M_j})_{j \in J}$$

to transport the Lie group structure of the weak direct product to the left hand side. Using Theorems A and B, we shall see that  $C_c^k(M, H)$  is complete whenever  $H$  and its modelling space  $E$  are complete (Proposition 6.7). Likewise, gauge groups and full symmetry groups of principal bundles (as considered in [48] and [53]) are complete if the structure group  $H$  and its modelling space are complete (see Remark 6.8 for more details).

**Linear Lie groups.** We can also prove completeness for some unit groups of topological algebras.<sup>10</sup> Consider an ascending sequence  $A_1 \subseteq A_2 \subseteq \dots$  of unital Banach algebras, such that all inclusion maps  $A_n \rightarrow A_{n+1}$  are continuous homomorphisms of unital algebras. Endow  $A := \bigcup_{n \in \mathbb{N}} A_n$  with the unital algebra structure turning each inclusion map  $A_n \rightarrow A$  into a homomorphism of unital algebras. Then the locally convex direct limit topology makes  $A$  a topological algebra and product sets are large in  $A^\times = \bigcup_{n \in \mathbb{N}} A_n^\times$  (see [26, Proposition 12.1 (a) and (c)]).<sup>11</sup> With Theorem A, we deduce that  $A^\times$  (like

<sup>10</sup>If  $A$  is a unital algebra, we write  $A^\times$  for its *unit group* of all invertible elements.

<sup>11</sup>Compare also [17] for general information on such algebras.

each  $A_n^\times$ ) is complete whenever the direct sequence  $A_1 \subseteq A_2 \subseteq \dots$  is strict.

**Ascending unions of Banach-Lie groups.** Beyond unit groups of Banach algebras, let us consider an ascending sequence  $G_1 \subseteq G_2 \subseteq \dots$  of Banach-Lie groups over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  such that each inclusion map  $G_n \rightarrow G_{n+1}$  is a  $\mathbb{K}$ -analytic group homomorphism. In [15, Theorem C], conditions were spelled out which ensure that  $G = \bigcup_{n \in \mathbb{N}} G_n$  can be made a  $\mathbb{K}$ -analytic Lie group modelled on the locally convex direct limit of the respective Lie algebras. We show that product sets are large in the Lie groups  $G$  constructed in loc. cit., whence the given topology on  $G$  makes it the direct limit topological group  $G = \varinjlim G_n$  (see Proposition 7.3). Using our Theorem A, we deduce that  $G = \bigcup_{n \in \mathbb{N}} G_n$  (as before) is complete whenever the direct sequence  $G_1 \subseteq G_2 \subseteq \dots$  is strict (see Proposition 7.3).

For a concrete example, let  $(F_n, \|\cdot\|_n)_{n \in \mathbb{N}}$  be a sequence of Banach spaces. Write  $\mathcal{L}(F)$  for the Banach algebra of bounded operators  $S: F \rightarrow F$  for a Banach space  $F$ , endowed with the operator norm. We equip  $E_n := F_1 \oplus \dots \oplus F_n$  with the maximum norm and identify

$$\mathrm{GL}(E_n) := \mathcal{L}(E_n)^\times$$

with the subgroup  $\mathrm{GL}(E_n) \times \{\mathrm{id}_{F_{n+1}}\}$  of  $\mathrm{GL}(E_{n+1})$ . Using [16, Theorem A], it was shown in [14, Theorem 39] that

$$\mathrm{GL}((F_n)_{n \in \mathbb{N}}) := \bigcup_{n \in \mathbb{N}} \mathrm{GL}(E_n)$$

can be made a Lie group.<sup>12</sup> As the direct sequence  $\mathrm{GL}(E_1) \subseteq \mathrm{GL}(E_2) \subseteq \dots$  is strict, the preceding reasoning shows that  $\mathrm{GL}((F_n)_{n \in \mathbb{N}})$  is complete.

**Lie groups modelled on Silva spaces.** After this research was completed, it was shown in [13] that a topological group is complete whenever its underlying topological space is locally  $k_\omega$  (by a modification of the arguments from [46] for abelian topological groups). As a consequence, every Lie group modelled on a Silva space<sup>13</sup> is complete (see [13]). This entails that direct limits  $\bigcup_{n \in \mathbb{N}} G_n$  of finite-dimensional Lie groups (or locally compact groups)

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<sup>12</sup>In loc. cit.,  $\mathrm{GL}((F_n)_{n \in \mathbb{N}})$  is denoted by  $\mathrm{GL}(E)$ , with  $E := \bigcup_{n \in \mathbb{N}} E_n$ .

<sup>13</sup>A locally convex space is called a *Silva space* or (DFS)-space if it is a locally convex direct limit  $\varinjlim E_n$  for an ascending sequence  $E_1 \subseteq E_2 \subseteq \dots$  of Banach spaces, such that all inclusion maps  $E_n \rightarrow E_{n+1}$  are compact operators.



$G_1 \subseteq G_2 \subseteq \dots$  are always complete (no matter whether the direct sequence is strict or not). It also shows that the Lie group  $\text{Diff}^\omega(M)$  of real-analytic diffeomorphisms is complete for each compact real-analytic manifold  $M$ , as well as the Lie group  $C^\omega(M, H)$  of all real-analytic  $H$ -valued mappings on the latter, for each finite-dimensional Lie group  $H$  (see [13]).

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## 1 Preliminaries and notation

We write  $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . Topological groups and locally convex (real topological vector) spaces shall not be assumed Hausdorff, unless we say so explicitly. If  $f: X \rightarrow Y$  is a function between metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , we define

$$\text{Lip}(f) := \sup \left\{ \frac{d_Y(f(x), f(y))}{d_X(x, y)} : x \neq y \in X \right\} \in [0, \infty]$$

and call  $f$  *Lipschitz* if  $\text{Lip}(f) < \infty$ . If each point  $x \in X$  has a neighbourhood  $V \subseteq X$  such that  $f|_V: V \rightarrow Y$  is Lipschitz (with respect to the metric  $d_X|_{V \times V}$  induced on  $V$ ), then  $f$  is called *locally Lipschitz*. If  $(E, \|\cdot\|)$  is a Banach space, we write  $\text{GL}(E)$  for the group of continuous automorphisms of the vector space  $E$ . For  $x \in E$  and  $r > 0$ , we write  $B_r^E(x) := \{y \in E: \|y - x\| < r\}$  and  $\overline{B}_r^E(x) := \{y \in E: \|y - x\| \leq r\}$ . If  $q$  is a continuous seminorm on a locally convex space  $E$ , we write  $\overline{B}_r^q(0) := \{x \in E: q(x) \leq r\}$  for  $r > 0$ .

Given normed spaces  $E$  and  $F$ , we write  $\mathcal{L}(E, F)$  for the space of continuous linear mappings from  $E$  to  $F$ , endowed with the operator norm  $\|\cdot\|_{op}$ . As usual,  $\mathcal{L}(E) := \mathcal{L}(E, E)$ .

Our conventions and notation concerning manifolds and Lie groups (which are modelled on Hausdorff locally convex spaces) are compiled in Appendix A. In particular, every Lie group is a topological group in its given topology (as smooth maps are always continuous in the infinite-dimensional calculus we are using).

Recall that a net  $(x_\alpha)_{\alpha \in A}$  in a topological group  $G$  is called a left Cauchy net

if, for each identity neighbourhood  $U \subseteq G$ , there exists  $\gamma \in A$  such that

$$x_\beta^{-1} x_\alpha \in U \quad \text{for all } \alpha, \beta \geq \gamma.$$

If every left Cauchy net in  $G$  is convergent, then  $G$  is called *complete*; if every left Cauchy sequence in  $G$  is convergent, then  $G$  is *sequentially complete*.

**1.1** Many results concerning completeness of topological groups can be found in [47]. We mention useful facts:

- (a) If a topological group  $G$  is complete (resp., sequentially complete), then every closed subgroup  $H \subseteq G$  is complete in the induced topology.
- (b) For every family  $(G_j)_{j \in J}$  of topological groups which are complete (resp., sequentially complete), the direct product  $\prod_{j \in J} G_j$  is complete (resp., sequentially complete) in the product topology.
- (c) Let  $((G_j)_{j \in J}, (q_{i,j})_{i \leq j})$  be a projective system of Hausdorff topological groups  $G_j$  and continuous homomorphisms  $q_{i,j}: G_j \rightarrow G_i$  for  $i \leq j$  in  $J$  such that  $q_{i,i} = \text{id}_{G_i}$  and  $q_{i,j} \circ q_{j,k} = q_{i,k}$  whenever  $i \leq j \leq k$ . If each  $G_i$  is complete (resp., sequentially complete), then also the projective limit  $\varprojlim G_j$  is complete (resp., sequentially complete), as it can be realized as the closed subgroup

$$\left\{ (x_j)_{j \in J} \in \prod_{j \in J} G_j : (\forall i \leq j) \ x_i = q_{i,j}(x_j) \right\}$$

of the direct product, endowed with the induced topology.

- (d) Completeness is an extension property: If  $G$  is a topological group and  $N \subseteq G$  a normal subgroup such that both  $N$  and  $G/N$  are complete, then also  $G$  is complete (cf. [47, Theorem 12.3 (a)]).

If  $G$  is a topological group,  $H \subseteq G$  a subgroup and  $N \subseteq G$  a normal subgroup, we say that  $G$  is the (internal) *semidirect product* of  $N$  and  $H$  as a topological group if the product map

$$N \times H \rightarrow G, \quad (x, y) \mapsto xy$$

is a homeomorphism. Since  $q: G \rightarrow H, xy \mapsto y$  is a quotient homomorphism with kernel  $N$ , the following holds as special case of (d):

- (e) Let  $G$  be a topological group which, as a topological group, is the internal semidirect product of a normal subgroup  $N$  and a subgroup  $H$ . If  $N$  and  $H$  are complete, then also  $G$  is complete ([47, Proposition 12.5 (a)]).

The following slight generalization of 1.1 (c) is useful.

**Lemma 1.2** *Let  $G$  be a topological group whose underlying topological space is the projective limit of a projective system  $((X_j)_{j \in J}, (q_{i,j})_{i \leq j})$  of Hausdorff topological spaces  $X_j$ , with limit maps  $q_j: G \rightarrow X_j$  for  $j \in J$ . Assume that for each Cauchy net  $(x_\alpha)_{\alpha \in A}$  in  $G$ , the corresponding net  $(q_j(x_\alpha))_{\alpha \in A}$  converges in  $X_j$ , for each  $j \in J$ . Then  $G$  is complete.*

**Proof.** We may assume that

$$G = \left\{ (x_j)_{j \in J} \in \prod_{j \in J} X_j : (\forall i \leq j) \ x_i = q_{i,j}(x_j) \right\}$$

and  $q_i((x_j)_{j \in J}) = x_i$  for all  $i \in I$  and  $x = (x_j)_{j \in J} \in G$ . For  $j \in J$ , let  $y_j \in X_j$  be the limit of  $(q_j(x_\alpha))_{\alpha \in A}$ . For  $i, j \in I$  with  $i \leq j$ , the net

$$(q_i(x_\alpha))_{\alpha \in A} = (q_{i,j}(q_j(x_\alpha)))_{\alpha \in A}$$

in  $X_i$  converges both to  $y_i$  and  $q_{i,j}(y_j)$ . Since  $X_i$  is Hausdorff,  $y_i = q_{i,j}(y_j)$  follows. Hence  $y := (y_j)_{j \in J} \in G$  and as  $q_j(x_\alpha) \rightarrow y_j = q_j(y)$  for all  $j \in J$ , the net  $(x_\alpha)_{\alpha \in A}$  converges to  $y$ .  $\square$

**1.3** If  $(G_j)_{j \in J}$  is a family of topological groups, let

$$\square_{j \in J} G_j \subseteq \prod_{j \in J} G_j$$

be the subgroup of all  $(x_j)_{j \in J} \in \prod_{j \in J} G_j$  such that  $x_j = e$  for all but finitely many  $j$ . Consider the sets

$$\square_{j \in J} U_j := \prod_{j \in J} U_j \cap \square_{j \in J} G_j,$$

for  $(U_j)_{j \in J}$  ranging through the families of open subsets  $U_j \subseteq G_j$  such that  $e \in U_j$  for all but finitely many  $j$ . The latter sets form a basis for a topology on  $\square_{j \in J} G_j$  making it a topological group (called the *box topology*). When endowed with this topology,  $\square_{j \in J} G_j$  is called the *small box product* of the

family  $(G_j)_{j \in J}$  (see [4] for further details).

If each  $G_j$  is a Lie group modelled on a Hasudorff locally convex space, then also  $\square_{j \in J} G_j$  is a Lie group in a natural way (see [28]);<sup>14</sup> it is modelled on the small box product  $\square_{j \in J} E_j$ . If, instead, we use the locally convex direct sum as the modelling space, then the group  $\square_{j \in J} G_j$  can be made a Lie group as well, called the *weak direct product* of the family and denoted by  $\bigoplus_{j \in J} G_j$  in this article (see [22], where the notation  $\prod_{j \in J}^* G_j$  is used). The two possible modelling spaces (and the two Lie groups) coincide if  $J$  is countable. When dealing with  $\bigoplus_{j \in J} G_j$ , we write  $\bigoplus_{j \in J} U_j$  instead of  $\square_{j \in J} U_j$ .

## 2 Deducing completeness from completeness of a subgroup

The following lemma is essential for the proof of Theorem A. Because the (known) fact (d) from 1.1 (which shall be used in Section 6) happens to be an immediate consequence of the lemma, we also record this proof, for the reader's convenience.

**Lemma 2.1** *Let  $(x_\alpha)_{\alpha \in A}$  be a left Cauchy net in a topological group  $G$  and  $H \subseteq G$  be a subgroup which is a complete topological group in the induced topology. Assume that, for each  $\alpha \in A$  and identity neighbourhood  $W \subseteq G$ , there exists  $\beta \geq \alpha$  such that*

$$x_\beta \in HW. \quad (3)$$

*Then  $(x_\alpha)_{\alpha \in A}$  converges in  $G$ , to some  $y \in H$ .*

**Proof.** Let  $\mathcal{U}$  be the set of all identity neighborhoods in  $G$ . By hypothesis,

$$A_W := \{\alpha \in A : x_\alpha \in HW\}$$

is cofinal in  $A$  for all  $W \in \mathcal{U}$  and thus

$$M := \{(W, \alpha) \in \mathcal{U} \times A : \alpha \in A_W\}$$

becomes a directed set if we write  $(W_1, \alpha_1) \leq (W_2, \alpha_2)$  if and only if  $W_2 \subseteq W_1$  and  $\alpha_1 \leq \alpha_2$ . For  $a = (W, \alpha) \in M$ , pick  $y_a \in H$  and  $w_a \in W$  such that

$$x_\alpha = y_a w_a. \quad (4)$$

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<sup>14</sup>In [28], small box products are called weak direct products, in contrast to the conventions in the current article.

Then  $(y_a)_{a \in M}$  is a left Cauchy net in  $H$ . In fact, if  $U$  is an identity neighborhood in  $H$ , we find  $Q \in \mathcal{U}$  such that  $U = H \cap Q$ . Let  $P \in \mathcal{U}$  such that  $PPP^{-1} \subseteq Q$  and  $\gamma \in A$  such that

$$x_\beta^{-1}x_\alpha \in P \quad \text{for all } \alpha, \beta \geq \gamma.$$

We may assume that  $\gamma \in A_P$ . For all  $a, b \geq (P, \gamma)$  in  $M$ , say  $a = (W, \alpha)$  and  $b = (V, \beta)$ , we have  $V, W \subseteq P$  and hence

$$y_b^{-1}y_a = w_b x_\beta^{-1} x_\alpha w_a^{-1} \in VPW^{-1} \subseteq PPP^{-1} \subseteq Q.$$

Thus  $y_b^{-1}y_a \in H \cap Q = U$ .

Let  $y$  be the limit of  $(y_a)_{a \in M}$  in  $H$ . Then  $y_a \rightarrow y$  also in  $G$ . Given  $W \in \mathcal{U}$ , let  $\alpha \in A_W$ . Since  $w_a \in W$  for  $a \geq (W, \alpha)$ , the net  $(w_a)_{a \in M}$  converges to  $e$  in  $G$ . Using (4), we deduce that the subnet

$$(x_\alpha)_{(W, \alpha) \in M}$$

of  $(x_\alpha)_{\alpha \in A}$  (and hence also the Cauchy net  $(x_\alpha)_{\alpha \in A}$ ) converges to  $y$ .  $\square$

**Proof for 1.1 (d).** Let  $G$  be a topological group and  $N \subseteq G$  be a normal subgroup such that both  $N$  and  $G/N$  are complete. Let  $q: G \rightarrow G/N$ ,  $x \mapsto Nx$  be the canonical quotient map. If  $(x_\alpha)_{\alpha \in A}$  is a left Cauchy net in  $G$ , then  $(q(x_\alpha))_{\alpha \in A}$  is a left Cauchy net in  $G/N$  and hence convergent to  $Nz$  for some  $z \in G$ . Let  $y_\alpha := z^{-1}x_\alpha$ . Then  $(y_\alpha)_{\alpha \in A}$  is a left Cauchy net in  $G$ , as

$$y_\beta^{-1}y_\alpha = x_\beta^{-1}zz^{-1}x_\alpha = x_\beta^{-1}x_\alpha$$

for all  $\alpha, \beta \in A$ . Since

$$q(y_\beta) = q(z^{-1}x_\beta) = q(z)^{-1}q(x_\beta) \rightarrow q(z)^{-1}q(z) = e,$$

for each identity neighbourhood  $W \subseteq G$  we find  $\alpha_W \in A$  such that  $q(y_\beta) \in q(W)$  for all  $\beta \geq \alpha_W$  and hence

$$y_\beta \in NW$$

for all  $\beta \geq \alpha_W$ . Given  $\alpha \in A$ , we can find  $\beta \in A$  such that  $\beta \geq \alpha$  and  $\beta \geq \alpha_W$ , since  $(A, \leq)$  is directed. Since  $y_\beta \in NW$ , the hypotheses of Lemma 2.1 are satisfied. Hence, the net  $(y_\alpha)_{\alpha \in A}$  converges to some  $y \in N$ . As a consequence,  $x_\alpha = zy_\alpha \rightarrow zy$ .  $\square$

### 3 Completeness of strict direct limits

In this section, we prove Theorem A.

**Lemma 3.1** *Let  $G$  be a group,  $2 \leq n \in \mathbb{N}$  and  $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$  be subgroups. For  $j \in \{1, \dots, n\}$ , let  $W_j$  be a subset of  $G_j$ . Then*

$$G_1 \cap (W_1 W_2 \cdots W_n) = G_1 \cap (W_1 \cdots W_{n-1} (G_{n-1} \cap W_n)). \quad (5)$$

**Proof.** We show by induction on  $n$  that the left hand side of (5) is a subset of the right hand side (the other inclusion is trivial). If  $n = 2$  and  $x \in G_1 \cap W_1 W_2$ , then  $x = w_1 w_2$  with  $w_1 \in W_1$  and  $w_2 \in W_2$ . Since  $W_1 \subseteq G_1$ , we have  $w_1 \in G_1$  and thus  $w_2 = w_1^{-1} x \in G_1 \cap W_2$ .

If  $n > 2$  and the assertion holds for  $n - 1$ , let  $x \in G_1 \cap (W_1 \cdots W_n)$ . Write  $x = w_1 w_2 \cdots w_n$  with  $w_j \in W_j$  for  $j \in \{1, \dots, n\}$ . Then  $w_2 \cdots w_n = w_1^{-1} x \in G_1 \cap (W_2 \cdots W_n) \subseteq G_2 \cap (W_2 \cdots W_n)$  and thus  $w_n \in G_{n-1} \cap W_n$ , by the inductive hypothesis.  $\square$

**Lemma 3.2** *Assume that  $G$  is the direct limit topological group of a direct sequence  $G_1 \subseteq G_2 \subseteq \cdots$  of topological groups and product sets are large in  $G$ . Then every identity neighbourhood of  $G$  contains a product set*

$$\bigcup_{n \in \mathbb{N}} W_1 W_2 \cdots W_n$$

for suitable identity neighbourhoods  $W_n \subseteq G_n$ .

**Proof.** If  $V_0$  is an identity neighbourhood in  $G$ , there exist identity neighbourhoods  $V_n \subseteq G$  for  $n \in \mathbb{N}$  such that  $V_n V_n \subseteq V_{n-1}$ . Then  $W_n := G_n \cap V_n$  is an identity neighbourhood in  $G_n$  and

$$V_1 V_2 \cdots V_n \subseteq V_0 \quad \text{for all } n \in \mathbb{N}$$

implies that  $\bigcup_{n \in \mathbb{N}} W_1 \cdots W_n \subseteq \bigcup_{n \in \mathbb{N}} V_1 \cdots V_n \subseteq V_0$ .  $\square$

**Proof of Theorem A.** (a) To see that  $G$  induces the given topology on  $G_n$ , we may assume that  $n = 1$ . Let  $V_1 \subseteq G_1$  be an identity neighbourhood. There exists an identity neighbourhood  $W_1 \subseteq G_1$  such that  $W_1 W_1 \subseteq V_1$ . Recursively, for  $m \geq 2$  find an identity neighbourhood  $V_m \subseteq G_m$  such that

$$G_{m-1} \cap V_m = W_{m-1}$$

(which is possible as  $G_m$  induces the given topology on  $G_{m-1}$ ) and an identity neighbourhood  $W_m \subseteq G_m$  such that

$$W_m W_m \subseteq V_m.$$

Then

$$G_1 \cap (W_1 W_2 \cdots W_m) \subseteq G_1 \cap (W_1 \cdots W_{j-1} V_j) \quad (6)$$

for all  $j \in \{m, m-1, \dots, 1\}$ , by induction: If  $j = m$ , we have  $W_m \subseteq V_m$  and the assertion holds. If  $j \in \{2, \dots, m\}$  and the assertion holds for  $j$ , then

$$\begin{aligned} G_1 \cap (W_1 W_2 \cdots W_m) &\subseteq G_1 \cap (W_1 \cdots W_{j-1} V_j) \\ &= G_1 \cap (W_1 \cdots W_{j-1} (G_{j-1} \cap V_j)) \\ &= G_1 \cap (W_1 \cdots W_{j-1} W_{j-1}) \\ &\subseteq G_1 \cap (W_1 \cdots W_{j-2} V_{j-1}) \end{aligned}$$

using the inductive hypothesis, Lemma 3.1, the identity  $G_{j-1} \cap V_j = W_{j-1}$  and the inclusion  $W_{j-1} W_{j-1} \subseteq V_{j-1}$ . Taking  $j = 1$ , we deduce that

$$G_1 \cap (W_1 W_2 \cdots W_m) \subseteq V_1$$

for all  $m \in \mathbb{N}$  and hence  $G_1 \cap W \subseteq V_1$  if we define

$$W := \bigcup_{m \in \mathbb{N}} W_1 W_2 \cdots W_m.$$

As we assume that product sets are large in  $G$ , the set  $W$  is an identity neighborhood in  $G$ . Since  $G_1 \cap W \subseteq V_1$ , the group topology  $\mathcal{T}$  induced by  $G$  on  $G_1$  is finer than the given topology  $\mathcal{O}_1$  on  $G_1$  and hence coincides with it (noting that  $\mathcal{T} \subseteq \mathcal{O}_1$  as the inclusion map  $(G_1, \mathcal{O}_1) \rightarrow G$  is continuous).

(b) If each  $G_n$  is complete, let  $(x_\alpha)_{\alpha \in A}$  be a left Cauchy net in  $G$ . Let  $\mathcal{U}$  be the set of all identity neighborhoods in  $G$ . We claim that there exists  $m \in \mathbb{N}$  such that, for each  $\alpha \in A$  and  $W \in \mathcal{U}$ , there exists  $\beta \geq \alpha$  such that

$$x_\beta \in G_m W.$$

If this is true, then  $(x_\alpha)_{\alpha \in A}$  converges in  $G$ , by Lemma 2.1, using that  $G$  induces the given complete group topology on  $G_m$ , by (a).

To prove the claim, suppose it was wrong. Then, for each  $m \in \mathbb{N}$ , there exist  $\alpha_m \in A$  and  $W_m \in \mathcal{U}$  such that

$$x_\alpha \notin G_m W_m \quad \text{for all } \alpha \geq \alpha_m. \quad (7)$$

After shrinking  $W_m$  if necessary, we may assume that

$$W_m = \bigcup_{n \in \mathbb{N}} W_1^{(m)} \cdots W_n^{(m)}$$

with identity neighbourhoods  $W_n^{(m)} \subseteq G_n$ , by Lemma 3.2. After shrinking  $W_n^{(2)}, W_n^{(3)}, \dots$ , we may assume that

$$W_n^{(m+1)} \subseteq W_n^{(m)} \quad \text{for all } n, m \in \mathbb{N}. \quad (8)$$

Since product sets are large in  $G$ , the set

$$W := \bigcup_{n \in \mathbb{N}} W_1^{(1)} \cdots W_n^{(n)}$$

is an identity neighbourhood in  $G$ . By (8), we have

$$\bigcup_{n > m} W_{m+1}^{(m+1)} \cdots W_n^{(n)} \subseteq \bigcup_{n > m} W_{m+1}^{(m)} \cdots W_n^{(m)} \subseteq \bigcup_{n \in \mathbb{N}} W_1^{(m)} \cdots W_n^{(m)} = W_m.$$

Using that  $W_1^{(1)} \cdots W_n^{(n)} \subseteq G_m$  for  $n \in \{1, \dots, m\}$ , we deduce that

$$G_m W = G_m \bigcup_{n > m} W_{m+1}^{(m+1)} \cdots W_n^{(n)} \subseteq G_m W_m$$

and thus

$$G_m W \subseteq G_m W_m \quad \text{for all } m \in \mathbb{N}. \quad (9)$$

By definition of a Cauchy net, we find  $\gamma \in A$  such that

$$x_\alpha^{-1} x_\beta \in W \quad (10)$$

for all  $\alpha, \beta \geq \gamma$ . Now  $x_\gamma \in G_{m_0}$  for some  $m_0 \in \mathbb{N}$ . Since  $A$  is directed, we find  $\alpha \in A$  such that  $\alpha \geq \gamma$  and  $\alpha \geq \alpha_{m_0}$ . Using (10), we obtain

$$x_\alpha = x_\gamma (x_\gamma^{-1} x_\alpha) \in G_{m_0} W.$$

But  $x_\alpha \notin G_{m_0} W_{m_0}$  by (7) and thus  $x_\alpha \notin G_{m_0} W$  (by (9)), which is absurd.  $\square$



## 4 Completeness of weak direct products

The following lemma will enable us to reduce the completeness of weak direct products (and box products) to that of direct products.

**Lemma 4.1** *Let  $P$  be a complete (resp., sequentially complete) topological group and  $G$  be a subgroup of  $P$ , endowed with a topology  $\mathcal{O}$  which is finer than the topology  $\mathcal{T}$  induced by  $P$  on  $G$ . Assume that, for each  $x \in P$  such that  $x \notin G$ , there exists a closed subset  $L$  in  $P$  such that*

- (i)  $G \cap L$  is an identity neighborhood in  $(G, \mathcal{O})$ ; and
- (ii)  $G \cap xL = \emptyset$ .

Moreover, assume that

- (iii) The closures  $\overline{V}$  in  $(G, \mathcal{T})$  form a basis of identity neighbourhoods in  $(G, \mathcal{O})$ , for  $V$  in the set of identity neighbourhoods in  $(G, \mathcal{O})$ .

Then also  $(G, \mathcal{O})$  is complete (resp., sequentially complete).

**Proof.** Assume that  $P$  is complete. If  $(x_\alpha)_{\alpha \in A}$  is a left Cauchy net in  $(G, \mathcal{O})$ , then it also is a left Cauchy net in  $P$ , whence

$$x_\alpha \rightarrow x \text{ in } P$$

for some  $x \in P$ . Then  $x \in G$ , since otherwise we obtain a contradiction: Let  $L \subseteq P$  be a closed subset such that  $G \cap L$  is an identity neighbourhood in  $(G, \mathcal{O})$  and  $G \cap xL = \emptyset$ . Let  $\alpha_0 \in A$  such that

$$x_\alpha^{-1}x_\beta \in G \cap L$$

for all  $\alpha, \beta \geq \alpha_0$ . Considering  $x_\alpha^{-1}x_\beta$  as elements of  $P$  and passing to the limit in  $\alpha$ , we obtain

$$x^{-1}x_\beta \in L$$

for all  $\beta \geq \alpha_0$ , whence  $x_\beta \in G \cap xL = \emptyset$ , which is absurd. Thus  $x \in G$ .

Let  $W$  be an identity neighborhood in  $(G, \mathcal{O})$ . By hypothesis (iii), we find an identity neighbourhood  $V$  in  $(G, \mathcal{O})$  such that

$$G \cap \overline{V} \subseteq W,$$

where  $\overline{V}$  is the closure of  $V$  in  $P$ . There exists  $\alpha_0 \in A$  such that

$$x_\alpha^{-1}x_\beta \in V \quad \text{for all } \alpha, \beta \geq \alpha_0.$$

Considering  $x_\alpha^{-1}x_\beta$  as an element of  $P$  and passing to the limit in  $\alpha$ , we deduce that

$$x^{-1}x_\beta \in \overline{V}$$

for all  $\beta \geq \alpha_0$ , whence  $x^{-1}x_\beta \in G \cap \overline{V} \subseteq W$  and thus  $x_\beta \in xW$ . Thus  $x_\beta \rightarrow x$  in  $(G, \mathcal{O})$ .

If  $P$  is sequentially complete, then the proof is identical with  $A := \mathbb{N}$ .  $\square$

**Example 4.2** Let  $(G_j)_{j \in J}$  be a family of topological groups  $G_j$  which are complete (resp., sequentially complete). Then also the small box product  $G := \square_{j \in J} G_j$  is complete (resp., sequentially complete).

To see this, let us check Lemma 4.1 can be used with  $P := \prod_{j \in J} G_j$ . If  $x = (x_j)_{j \in J} \in P \setminus G$ , then

$$I := \{j \in J : x_j \neq e\}$$

is an infinite set. For each  $j \in I$ , let  $U_j \subseteq G_j$  be a closed identity neighbourhood such that  $x_j^{-1} \notin U_j$ . For  $j \in J \setminus I$ , let  $U_j := G_j$ . Then  $L := \prod_{j \in J} U_j$  is closed in  $P$  and  $G \cap L = \square_{j \in J} U_j$  is an identity neighbourhood such that  $xL \cap G = \emptyset$  as each  $y = (y_j)_{j \in J} \in xL$  satisfies  $y_j \neq e$  for all  $j \in I$ . Finally, let  $W \subseteq G$  be an identity neighbourhood. Then  $W$  contains a box  $V := \square_{j \in J} V_j$  with closed identity neighbourhoods  $V_j \subseteq G_j$ . The closure  $\overline{V}$  of  $V$  in  $P$  is  $\prod_{j \in J} V_j$ , whence  $G \cap \overline{V} = V \subseteq W$ .

**Proof of Theorem B.** To verify the theorem, let us check that Lemma 4.1 can be applied with  $P := \prod_{j \in J} G_j$ . If  $x = (x_j)_{j \in J} \in P \setminus G$ , then

$$I := \{j \in J : x_j \neq e\}$$

is an infinite set. For each  $j \in I$ , let  $U_j \subseteq G_j$  be a closed identity neighbourhood such that  $x_j^{-1} \notin U_j$ . For  $j \in J \setminus I$ , let  $U_j := G_j$ . Then  $L := \prod_{j \in J} U_j$  is closed in  $P$  and  $G \cap L = \square_{j \in J} U_j$  is an identity neighbourhood (as the topology on  $G$  is finer than the box topology) such that  $xL \cap G = \emptyset$ . Thus  $L$  satisfies the conditions (i) and (ii) in Lemma 4.1.

Next, let  $S \subseteq G$  be an identity neighbourhood. For  $j \in J$ , let  $E_j$  be the

locally convex space on which  $G_j$  is modelled. Then  $S$  contains an identity neighbourhood of the form

$$W = \phi^{-1}(Q)$$

for a diffeomorphism  $\phi: U \rightarrow V$  and a 0-neighbourhood  $Q \subseteq V$ , where diffeomorphisms  $\phi_j: U_j \rightarrow V_j$  from open identity neighbourhoods  $U_j \subseteq G_j$  onto open 0-neighbourhoods  $V_j \subseteq E_j$  are used to define  $U := \bigoplus_{j \in J} U_j$ ,  $V := \bigoplus_{j \in J} V_j$  and

$$\phi := \bigoplus_{j \in J} \phi_j: U \rightarrow V.$$

For each  $j \in J$ , the topological group  $G_j$  has a closed identity neighbourhood  $K_j$  such that  $K_j \subseteq U_j$ . Set  $L_j := \phi_j(K_j)$ . After shrinking  $Q$  (and  $W = \phi^{-1}(Q)$ ), we may assume that

$$W \subseteq \bigoplus_{j \in J} K_j =: K$$

and thus  $Q \subseteq \prod_{j \in J} L_j =: L$ . After shrinking  $Q$  further if necessary, we may also assume that

$$Q = \overline{B}_r^q(0)$$

for a continuous seminorm  $q$  on  $\bigoplus_{j \in J} E_j$ , and we may assume that  $q$  is of the form

$$q(x) = \sum_{j \in J} q_j(x_j) \text{ for all } x = (x_j)_{j \in J} \in \bigoplus_{j \in J} E_j,$$

for certain continuous seminorms  $q_j$  on  $E_j$ . Then the closure  $\overline{Q}$  of  $Q$  in  $\prod_{j \in J} E_j$  is the set  $C$  of all  $(x_j)_{j \in J} \in \prod_{j \in J} E_j$  such that

$$\sum_{j \in J} q_j(x_j) \leq r$$

(where the sum means the supremum of all finite partial sums). To see this, let  $\Phi$  be the set of finite subsets of  $J$ . If  $x \in C$ , then  $\sum_{j \in F} x_j \in Q$  for each  $F \in \Phi$  (as  $\sum_{j \in F} q_j(x_j) \leq \sum_{j \in J} q_j(x_j) \leq r$ ) and thus  $x \in \overline{Q}$ .

Conversely, let  $x = (x_j)_{j \in J} \in \prod_{j \in J} E_j$  with  $x \notin C$ ; thus  $\sum_{j \in J} q_j(x_j) > r$ . There exists a finite subset  $F \subseteq J$  such that

$$\sum_{j \in F} q_j(x_j) > r.$$

Since  $\{(y_j)_{j \in J} \in \prod_{j \in J} E_j : \sum_{j \in F} q_j(y_j) > r\}$  is an open subset of  $\prod_{j \in J} E_j$  which has empty intersection with  $Q$ , we have  $x \notin \overline{Q}$ . Thus  $\overline{Q} = C$ .

By the preceding,  $(\bigoplus_{j \in J} E_j) \cap \overline{Q} = (\bigoplus_{j \in J} E_j) \cap C = Q$ . Since  $L$  is closed in  $\prod_{j \in J} E_j$  and contains  $Q$ , we have  $\overline{Q} \subseteq L$ . Now  $K = \prod_{j \in J} K_j$  is a closed subset of  $P = \prod_{j \in J} G_j$  and

$$\psi := \prod_{j \in J} (\phi_j|_{K_j}) : K \rightarrow L, \quad (x_j)_{j \in J} \mapsto (\phi_j(x_j))_{j \in J}$$

is a homeomorphism, whence  $\psi^{-1}(\overline{Q})$  is closed in  $K$  and hence also closed in  $P = \prod_{j \in J} G_j$ .

Let  $\overline{W}$  be the closure of  $W$  in  $P$ . By the preceding,  $\overline{W} \subseteq \psi^{-1}(\overline{Q})$  and thus  $\overline{W} = \psi^{-1}(\overline{Q})$ , as  $\psi$  is a homeomorphism. For  $x \in K$ , we have  $\psi(x) \in \bigoplus_{j \in J} E_j$  if and only if  $x \in \bigoplus_{j \in J} G_j$ . Hence

$$G \cap \overline{W} = G \cap \psi^{-1}(\overline{Q}) = \psi^{-1}\left(\left(\bigoplus_{j \in J} E_j\right) \cap \overline{Q}\right) = \psi^{-1}(Q) = W \subseteq S,$$

entailing that also hypothesis (iii) of Lemma 4.1 is satisfied.  $\square$

## 5 Completeness of diffeomorphism groups

The following proposition helps us to prove completeness of diffeomorphism groups.

**Proposition 5.1** *Let  $G$  be a topological group which is the projective limit of a projective sequence  $((G_n)_{n \in \mathbb{N}}, (q_{n,m})_{n \leq m})$  of topological Hausdorff groups. Assume that  $G_n$  admits a  $C^1$ -manifold structure modelled on a Banach space  $(E_n, \|\cdot\|_n)$  for each  $n \in \mathbb{N}$ , and assume that for each  $n \in \mathbb{N}$ , there exists  $m \geq n$  such that the map*

$$\mu_{n,m} : G_m \times G_m \rightarrow G_n, \quad (x, y) \mapsto q_{n,m}(xy) = q_{n,m}(x)q_{n,m}(y)$$

*is  $C^1$ . Then  $G$  is a complete topological group.*

**Proof.** For  $n \in \mathbb{N}$ , let  $q_n : G \rightarrow G_n$  be the limit map. Let  $(x_\alpha)_{\alpha \in A}$  be a left Cauchy net in  $G$ . By Lemma 1.2, it suffices to show that, for each  $n \in \mathbb{N}$ ,

the left Cauchy net  $(q_n(x_\alpha))_{\alpha \in A}$  converges in  $G_n$ . By hypothesis,  $\mu_{n,m}$  is  $C^1$  for some  $m \geq n$ . We write  $\|\cdot\|$  for the norm

$$E_n \times E_n \rightarrow [0, \infty[, \quad (x, y) \mapsto \max\{\|x\|_n, \|y\|_n\}.$$

Let  $\phi: U \rightarrow V$  be a  $C^1$ -diffeomorphism from an open identity neighbourhood  $U \subseteq G_m$  onto an open 0-neighbourhood  $V \subseteq E_m$  and  $\psi: P \rightarrow Q$  be a  $C^1$ -diffeomorphism from an open identity neighbourhood  $P \subseteq G_n$  onto an open 0-neighbourhood  $Q \subseteq E_n$ , such that  $\phi(e) = 0$  and  $\psi(e) = 0$ . After shrinking  $U$ , we may assume that

$$\mu_{n,m}(U \times U) \subseteq P,$$

enabling us to consider the  $C^1$ -map

$$f := \psi \circ \mu_{n,m} \circ (\phi^{-1} \times \phi^{-1}): V \times V \rightarrow Q, \quad (x, y) \mapsto \psi(q_{n,m}(\phi^{-1}(x)\phi^{-1}(y))).$$

After shrinking  $V$ , we may assume that  $f$  is Lipschitz (see Lemma A.7). Let  $L \geq 0$  be a Lipschitz constant for  $f$ . Then

$$\|f(x, y) - f(x, 0)\|_n \leq L\|(0, y)\| = L\|y\|_n$$

for all  $(x, y) \in V \times V$ . Let  $C$  be a closed 0-neighbourhood in  $E_n$  such that  $C \subseteq Q$ . Then  $q_n^{-1}(\psi^{-1}(C)) \cap q_m^{-1}(U)$  is an identity neighbourhood in  $G$ . There is  $\alpha_0 \in A$  such that

$$x_\beta^{-1}x_\alpha \in q_n^{-1}(\psi^{-1}(C)) \cap q_m^{-1}(U)$$

for all  $\alpha, \beta \geq \alpha_0$ . Thus

$$z_\alpha := x_{\alpha_0}^{-1}x_\alpha \in q_n^{-1}(\psi^{-1}(C)) \cap q_m^{-1}(U)$$

for all  $\alpha \geq \alpha_0$ , and  $(q_n(z_\alpha))_{\alpha \geq \alpha_0}$  is a left Cauchy net in  $G_n$ , noting that

$$z_\beta^{-1}z_\alpha = x_\beta^{-1}x_\alpha \in q_m^{-1}(U) \quad \text{for all } \alpha, \beta \geq \alpha_0.$$

If we can show that  $(q_n(z_\alpha))_{\alpha \geq \alpha_0}$  converges in  $G_n$ , then also the subnet  $(q_n(x_\alpha))_{\alpha \geq \alpha_0}$  of  $(q_n(x_\alpha))_{\alpha \in A}$  will converge, as  $q_n(x_\alpha) = q_n(x_{\alpha_0})q_n(z_\alpha)$ . Hence  $(q_n(x_\alpha))_{\alpha \in A}$  will converge. Since  $z_\alpha = z_\beta(z_\beta^{-1}z_\alpha)$  and  $z_\beta = z_\beta e$ , we see that

$$\psi(q_n(z_\alpha)) - \psi(q_n(z_\beta)) = f(q_m(z_\beta), q_m(z_\beta^{-1}z_\alpha)) - f(q_m(z_\beta), 0)$$

and thus

$$\|\psi(q_n(z_\alpha)) - \psi(q_n(z_\beta))\|_n \leq L\|q_m(z_\beta^{-1}z_\alpha)\|_n,$$

which can be made arbitrarily small for large  $\alpha, \beta$ . Hence  $(\psi(q_n(z_\alpha)))_{\alpha \geq \alpha_0}$  is a Cauchy net in  $(E_n, +)$  and thus convergent to some  $w \in E_n$ . Since  $w_\alpha := \psi(q_n(z_\alpha)) \in C$  for all  $\alpha \geq \alpha_0$  and  $C$  is closed in  $E$ , we deduce that  $w \in C \subseteq Q$ . As a consequence,  $q_n(z_\alpha) = \psi^{-1}(w_\alpha)$  converges to  $\psi^{-1}(w)$ .  $\square$

**Remark 5.2** (a) In particular, Proposition 5.1 shows that every strong (ILB)-Lie group (as in [44]) is complete.

(b) Since  $\text{Diff}(M)$  is a strong (ILB)-Lie group for each compact smooth manifold  $M$  without boundary (see [44]), we deduce from (a) that  $\text{Diff}(M)$  is complete.

(c) Let  $M$  be a  $\sigma$ -compact finite-dimensional smooth manifold (without boundary). If  $K \subseteq M$  is a compact subset, then

$$\text{Diff}_K(M) := \{\phi \in \text{Diff}_c(M) : (\forall x \in M \setminus K) \phi(x) = x\}$$

is a Lie subgroup of  $\text{Diff}_c(M)$ . Morse Theory (cf. [34]) provides a compact submanifold  $N \subseteq M$  with smooth boundary such that  $K$  is contained in the interior of  $N$ . Let  $N^*$  be the double of  $N$ , which is a compact smooth manifold without boundary obtained by glueing two copies of  $N$  along their boundary. Then  $\text{Diff}_K(M) \cong \text{Diff}_K(N^*)$  as a Lie group.<sup>15</sup> As  $\text{Diff}_K(N^*)$  is a closed subgroup of the complete topological group  $\text{Diff}(N^*)$ , we see that  $\text{Diff}_K(N^*)$  (and hence also  $\text{Diff}_K(M)$ ) is complete.

(d) Using Theorem A, we now obtain completeness of  $\text{Diff}_c(M)$  for  $\sigma$ -compact  $M$ , as described in the Introduction; applying Theorem B to an open subgroup, completeness of  $\text{Diff}_c(M)$  for paracompact  $M$  follows.

## 6 Completeness of mapping groups

We now discuss completeness of mapping groups and test function groups.

**6.1** If  $k \in \mathbb{N} \cup \{\infty\}$  and  $M$  is a  $C^k$ -manifold (possibly with boundary) modelled on a Hausdorff locally convex space, we let  $TM$  be the tangent bundle and recursively define  $T^{j+1}M := T(T^jM)$  for  $j \in \mathbb{N}$  such that  $j \leq k$ .

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<sup>15</sup>Using the double was stimulated by discussions in [49].

If  $f: M \rightarrow N$  is a  $C^k$ -map to another such manifold, we let  $Tf: TM \rightarrow TN$  be the tangent map and recursively set  $T^j f := T(T^{j-1}f): T^j M \rightarrow T^j N$  for all  $2 \leq j \in \mathbb{N}$  such that  $j \leq k$ . For convenience,  $T^0 M := M$  and  $T^0 f := f$ . We endow the set  $C^k(M, N)$  of all  $C^k$ -maps from  $M$  to  $N$  with the so-called *compact-open  $C^k$ -topology*, i.e., the initial topology with respect to the mappings

$$T^j: C^k(M, N) \rightarrow C(T^j M, T^j N), \quad f \mapsto T^j f$$

for  $j \in \mathbb{N}_0$  such that  $j \leq k$ , where  $C(T^j M, T^j N)$  is endowed with the compact-open topology (cf. [43]).

**6.2** If  $H$  is a Lie group, with multiplication  $\mu: H \times H \rightarrow H$ , then the tangent map  $T\mu: T(H \times H) \rightarrow TH$  makes  $TH$  a Lie group, if we identify  $T(H \times H)$  with  $TH \times TH$  in the usual way (see, e.g., [30]). Then  $C^k(M, H)$  is a group for  $M$  as before and  $k \in \mathbb{N}_0 \cup \{\infty\}$ , with pointwise product

$$fg := \mu \circ (f, g) \quad \text{for } f, g \in C^k(M, H).$$

If  $k \geq 1$ , then

$$T(fg) = T\mu \circ (Tf, Tg) = TfTg \tag{11}$$

is the product in  $C(TM, TH)$ , whence  $Tf$  is a group homomorphism and hence also  $T^j f$  for all  $j \in \mathbb{N}_0$  such that  $j \leq k$ . Since  $C(T^j M, T^j H)$  is a topological group for each  $j$  (see, e.g., [30, Lemma A.5.23 (a)]), we deduce that  $C^k(M, H)$  is a topological group (compare also [43] for this discussion).

Our first goal is to establish completeness properties for the topological groups  $C^k(M, H)$ . We show:

**Proposition 6.3** *Let  $H$  be a Lie group modelled on a Hausdorff locally convex space  $E$  and  $M$  be a finite-dimensional  $C^k$ -manifold (possibly with boundary) for some  $k \in \mathbb{N}_0 \cup \{\infty\}$ . If  $H$  and  $E$  are complete, then also the topological group  $C^k(M, H)$  is complete.*

The proof is based on two lemmas.

**Lemma 6.4** *Given  $k \in \mathbb{N}$ , let  $M$  be a finite-dimensional  $C^k$ -manifold (possibly with boundary) and  $N$  be a  $C^k$ -manifold. Then the map*

$$\theta: C^k(M, N) \rightarrow C(M, N) \times C^{k-1}(TM, TN), \quad f \mapsto (f, Tf)$$

*is a topological embedding with closed image. If  $N$  is a Lie group, then  $\theta$  is a homomorphism of groups.*

**Proof.** The final observation follows from (11).

It is clear that  $\theta$  is injective. Moreover, the topology  $\mathcal{O}$  making  $\theta$  a topological embedding is initial with respect to the inclusion map

$$T^0: C^k(M, N) \rightarrow C(M, N)$$

and  $T: C^k(M, N) \rightarrow C^{k-1}(TM, TN)$ . As the topology on  $C^{k-1}(TM, TN)$  is initial with respect to the maps  $T^j: C^{k-1}(TM, TN) \rightarrow C(T^{j+1}M, T^{j+1}N)$  for  $j \in \{0, \dots, k-1\}$ , we deduce with the well-known transitivity of initial topologies (see, e.g., [30, Lemma A.2.7]) that  $\mathcal{O}$  is initial with respect to  $T^0$  and the maps  $T^j \circ T: C^k(M, N) \rightarrow C(T^{j+1}M, T^{j+1}N)$  for  $j \in \{0, \dots, k-1\}$ . Hence  $\mathcal{O}$  coincides with the compact-open  $C^k$ -topology.

To see that  $\theta$  has closed image, let  $(f_\alpha, df_\alpha)_{\alpha \in A}$  be a net in the image of  $\theta$  which converges to some  $(f, g) \in C(M, N) \times C^{k-1}(TM, TN)$ . It now suffices to show that each  $x \in M$  has an open neighbourhood  $U$  such that  $f|_U$  is  $C^1$  and  $T(f|_U) = g|_{TU}$ ; then  $f$  is  $C^k$  and  $g = Tf$ , whence  $(f, g) = \theta(f)$  is in the image of  $\theta$ .

For  $x \in M$ , there is a chart  $\psi: U_\psi \rightarrow V_\psi \subseteq Y$  of  $N$  such that  $f(x) \in U_\psi$  (where  $Y$  is the modelling space of  $N$ ) and a chart  $\phi: U_\phi \rightarrow V_\phi \subseteq X$  of  $M$  with  $x \in U_\phi$  such that  $U_\phi$  has compact closure  $K := \overline{U_\phi}$  and  $f(\overline{U_\phi}) \subseteq U_\psi$  (where  $X$  is the modelling space of  $M$ ). As the compact-open  $C^k$ -topology on  $C^k(M, N)$  is finer than the compact-open topology, the set

$$W := \{h \in C^k(M, N) : h(K) \subseteq U_\psi\}$$

is an open neighbourhood of  $f$  in  $C^k(M, N)$ . Thus, we find  $\alpha_0 \in A$  such that  $f_\alpha \in W$  for all  $\alpha \in A$  such that  $\alpha \geq \alpha_0$ . For such  $\alpha$ , we can define

$$h_\alpha := \psi \circ f \circ \phi^{-1}: V_\phi \rightarrow Y.$$

Then  $h_\alpha \rightarrow \psi \circ f \circ \phi^{-1}$  and  $d(h_\alpha) \rightarrow d\psi \circ g|_{TU_\phi} \circ T\phi^{-1}$  uniformly on compact sets (see Lemmas A.5.3, A.5.5 and A.5.9 in [30]), entailing that  $h := \psi \circ f \circ \phi^{-1}$  is  $C^1$  with  $dh = d\psi \circ g|_{TU_\phi} \circ T\phi^{-1}$  (by [30, Lemma 1.4.16]).<sup>16</sup> Thus  $f|_{U_\phi}$  is  $C^1$  with  $T(f|_{U_\phi}) = g|_{TU_\phi}$ .  $\square$

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<sup>16</sup>To apply the lemma, give  $\{\alpha \in A : \alpha \geq \alpha_0\} \cup \{\infty\}$  the topology with  $\{\alpha\}$  for  $\alpha \in A$  with  $\alpha \geq \alpha_0$  and  $\{\alpha \in A : \alpha \geq \beta\} \cup \{\infty\}$  for  $\beta \in A$  with  $\beta \geq \alpha_0$  as a basis (which is a Hausdorff topology). The assertion follows with Lemma 1.4.16 and Proposition A.5.17 from [30].



**Lemma 6.5** *Let  $M$  and  $N$  be smooth manifolds (possibly with boundary), both modelled on Hausdorff locally convex spaces. Then*

$$C^\infty(M, N) = \varprojlim_{k \in \mathbb{N}_0} C^k(M, N)$$

*as a topological space, using the respective inclusion maps as the bonding maps and limit maps.*

**Proof.** Consider the standard realization  $P \subseteq \prod_{k \in \mathbb{N}_0} C^k(M, N)$  of the projective limit. As all bonding maps are the inclusion maps, it is the set of all sequences  $(f_k)_{k \in \mathbb{N}_0} \in \prod_{k \in \mathbb{N}_0} C^k(M, N)$  such that

$$f_j = f_k \quad \text{for all } j, k \in \mathbb{N}_0 \text{ such that } j \leq k.$$

Then  $f_0 = f_k$  for all  $k \in \mathbb{N}_0$  and thus  $f_0 \in C^\infty(M, N)$ , entailing that the map

$$\Phi: C^\infty(M, N) \rightarrow P, \quad f \mapsto (f)_{k \in \mathbb{N}_0}$$

is a bijection. The topology  $\mathcal{O}$  on  $C^\infty(M, N)$  making  $\Phi$  a homeomorphism is initial with respect to the compositions  $T^j \circ \pi_k \circ \Phi = T^j$  for  $k \in \mathbb{N}_0$  and  $j \in \mathbb{N}_0$  such that  $j \leq k$  (where  $\pi_k$  is the projection from the direct product onto its  $k$ th factor). It therefore coincides with the compact-open  $C^\infty$ -topology.  $\square$

**Proof of Proposition 6.3.** If we can show that  $C^k(M, H)$  is complete for each  $k \in \mathbb{N}_0$ , then also  $C^\infty(M, H)$  (which is the projective limit of the latter topological groups, by Lemma 6.5) will be complete. We proceed by induction. If  $k = 0$ , then  $C^0(M, H) = C(M, H)$  is complete since  $H$  is complete and  $M$ , being locally compact, is a  $k_{\mathbb{R}}$ -space<sup>17</sup> (see, e.g., [30, Lemma A.5.23 (d)]).

If  $k \in \mathbb{N}$  and the assertion holds for  $k - 1$  in place of  $k$ , then  $C^{k-1}(TM, TH)$  is complete as  $TM$  has finite dimension,  $TH \cong L(H) \rtimes H$  is complete (see 1.1 (e)) and also its modelling space  $E \times E$  is complete. Moreover,  $C(M, H)$  is complete. As, by Lemma 6.4, the topological group  $C^k(M, H)$  is isomorphic to a closed subgroup of the direct product  $C(M, H) \times C^{k-1}(M, TH)$  of complete groups, also  $C^k(M, H)$  is complete.  $\square$

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<sup>17</sup>Recall that a topological space  $X$  is called a  $k_{\mathbb{R}}$ -space if it is Hausdorff and functions  $f: X \rightarrow \mathbb{R}$  are continuous if and only if  $f|_K$  is continuous for each compact subset  $K \subseteq X$ .

**Remark 6.6** (a) If  $M$  is a compact smooth manifold and  $H$  a Lie group, then the topology on the Lie group  $C^k(M, H)$  (for  $k \in \mathbb{N}_0 \cup \{\infty\}$ ) coincides with the compact-open  $C^k$ -topology defined above (see [30]). Hence  $C^k(M, H)$  is complete whenever  $H$  and its modelling space are complete.

(b) If  $M$  is a finite-dimensional  $\sigma$ -compact smooth manifold and  $K \subseteq M$  a compact subset, then the topology on the Lie group

$$C_K^k(M, H) := \{\gamma \in C^k(M, H) : \text{supp}(\gamma) \subseteq K\}$$

is induced by the compact-open  $C^k$ -topology on  $C^k(M, H)$ . Since  $C_K^k(M, H)$  is a closed subgroup of  $C^k(M, H)$ , we deduce that  $C_K^k(M, H)$  is complete whenever  $H$  and its modelling space are complete.

**Proposition 6.7** *Let  $M$  be a paracompact finite-dimensional smooth manifold and  $H$  be a Lie group. If  $H$  and its modelling space are complete, then  $C_c^k(M, H)$  is complete for each  $k \in \mathbb{N}_0 \cup \{\infty\}$ .*

**Proof.** If  $M$  is  $\sigma$ -compact, we choose a sequence  $(K_n)_{n \in \mathbb{N}}$  of compact subsets of  $M$  such that  $M = \bigcup_{n \in \mathbb{N}} K_n$  and  $K_n \subseteq K_{n+1}^0$  for each  $n$ . Then

$$C_{K_1}^k(M, H) \subseteq C_{K_2}^k(M, H) \subseteq \dots$$

is a strict direct sequence of topological groups and product sets are large in  $C_c^k(M, H) = \bigcup_{n \in \mathbb{N}} C_{K_n}^k(M, H)$  as the product map

$$\bigoplus_{n \in \mathbb{N}} C_{K_n}^k(M, H) \rightarrow C_c^k(M, H), \quad (\gamma_1, \dots, \gamma_n, e, e, \dots) \mapsto \gamma_1 \gamma_2 \cdots \gamma_n$$

admits a smooth local section around  $e$  which takes  $e$  to  $e$  (see [26, Example 11.6 and Remark 11.5]). Since  $C_{K_n}^k(M, H)$  is complete for each  $n \in \mathbb{N}$  (see Remark 6.6 (b)), we deduce with Theorem A that  $C_c^k(M, H)$  is complete.

If  $M$  is merely paracompact, we let  $(M_j)_{j \in J}$  be the family of connected components of  $M$  (each of which is  $\sigma$ -compact). Then the map

$$\Phi: C_c^k(M, H) \rightarrow \bigoplus_{j \in J} C_c^k(M_j, H), \quad \gamma \mapsto (\gamma|_{M_j})_{j \in J}$$

is an isomorphism of groups and we give  $C_c^k(M, H)$  the smooth Lie group structure which turns  $\Phi$  into an isomorphism of Lie groups. As the weak direct product is complete by the first part of the proof and Theorem B, we see that also  $C_c^k(M, H)$  is complete.  $\square$

**Remark 6.8** Let  $H$  be a Lie group,  $M$  be a smooth manifold of dimension  $m \in \mathbb{N}$  and  $P \rightarrow M$  be a smooth principal bundle with structure group  $H$ .

(a) If  $M$  is  $\sigma$ -compact and the condition  $\text{SUB}_\oplus$  of [48] is satisfied,<sup>18</sup> then the gauge group  $\text{Gau}_c(P)$  of  $P$  is a Lie group which is isomorphic to a closed Lie subgroup of the weak direct product

$$\bigoplus_{n \in \mathbb{N}} C^\infty(K_n, H),$$

where  $(K_n)_{n \in \mathbb{N}}$  is a locally finite cover of  $M$  by  $m$ -dimensional compact smooth submanifolds  $K_n$  with boundary such that  $P$  is trivializable on some open neighbourhood of  $K_n$ . If  $H$  and its modelling space are complete, then  $C^\infty(K_n, H)$  is complete for each  $n \in \mathbb{N}$  (by Proposition 6.3), whence also the weak direct product is complete (by Theorem B) and hence also  $\text{Gau}_c(P)$ , being isomorphic to a closed subgroup of the latter as a topological group. Then the full group  $\text{Aut}_c(P)$  of compactly supported symmetries of  $P$  (which was made a Lie group in [48])<sup>19</sup> is complete, as it is an extension

$$\{e\} \rightarrow \text{Gau}_c(P) \rightarrow \text{Aut}_c(P) \rightarrow \text{Diff}_c(M)_P \rightarrow \{e\}$$

of Lie groups (and hence of topological groups) for some open subgroup  $\text{Diff}(M)_P \subseteq \text{Diff}_c(M)$ . Since  $\text{Diff}_c(M)$  is complete (as already observed) and also  $\text{Gau}_c(P)$  is complete, so is the extension  $\text{Aut}_c(P)$  (as recalled in 1.1 (d)).

(b) If  $M$  is paracompact and condition  $\text{SUB}_\oplus$  is satisfied by  $P|_C$  for each connected component  $C$  of  $M$ , let  $(M_j)_{j \in J}$  be the family of connected components of  $M$ . We can identify  $\text{Gau}_c(P)$  with the weak direct product  $\bigoplus_{j \in J} \text{Gau}_c(M_j)$  (whence it can be considered as a complete Lie group by (a) and Theorem B). Moreover,  $\text{Aut}_c(P)$  can be made a Lie group having the weak direct product  $\bigoplus_{j \in J} \text{Aut}_c(P|_{M_j})$  as an open subgroup. Hence  $\text{Aut}_c(P)$  is complete, using Theorem B.

## 7 Product sets in unions of Banach-Lie groups

We now discuss ascending unions of Banach-Lie groups.

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<sup>18</sup>This is automatic if  $H$  is *locally exponential* in the sense that  $H$  has a smooth exponential function which is a local  $C^\infty$ -diffeomorphism at 0.

<sup>19</sup>For compact  $M$ , the Lie group  $\text{Aut}(P)$  was already constructed in [53].

**7.1** Let  $G_1 \subseteq G_2 \subseteq \cdots$  be analytic Banach-Lie groups over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  such that the inclusion maps  $j_{n+1,n}: G_n \rightarrow G_{n+1}$  are  $\mathbb{K}$ -analytic group homomorphisms. Identifying the Banach-Lie algebra  $\mathfrak{g}_n := L(G_n)$  with the image of the map  $L(j_{n+1,n})$  in  $\mathfrak{g}_{n+1}$ , we can consider the ascending union

$$\mathfrak{g} := \bigcup_{n \in \mathbb{N}} \mathfrak{g}_n$$

and endow it with the locally convex direct limit topology. Give  $G := \bigcup_{n \in \mathbb{N}} G_n$  the unique group structure making each inclusion map  $G_n \rightarrow G$  a group homomorphism. Define

$$\exp_G: \mathfrak{g} \rightarrow G$$

piecewise via  $\exp_G(x) := \exp_{G_n}(x)$  if  $x \in \mathfrak{g}_n$ .

**7.2** (Dahmen's setting). *If, in the situation of 7.1,*

- (a)  $\mathfrak{g}$  is Hausdorff;
- (b) *There are norms  $\|\cdot\|_n$  on  $\mathfrak{g}_n$  defining its topology for  $n \in \mathbb{N}$ , such that the Lie bracket of each  $\mathfrak{g}_n$  and each inclusion map*

$$(\mathfrak{g}_n, \|\cdot\|_n) \rightarrow (\mathfrak{g}_{n+1}, \|\cdot\|_{n+1})$$

*has operator norm  $\leq 1$ ; and*

- (c)  $\exp_G$  is injective on some 0-neighbourhood,

*then  $G$  admits a unique  $\mathbb{K}$ -analytic Lie group structure such that  $P := \exp_G(Q)$  is open in  $G$  for some open 0-neighbourhood  $Q \subseteq \mathfrak{g}$  and  $\exp_G|_Q^P$  a diffeomorphism of  $\mathbb{K}$ -analytic manifolds. (See [15, Theorem C]).*

**Proposition 7.3** *Let  $G_1 \subseteq G_2 \subseteq \cdots$  be analytic Banach-Lie groups over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  such that the inclusion maps  $G_n \rightarrow G_{n+1}$  are  $\mathbb{K}$ -analytic group homomorphisms. Assume that Dahmen's conditions (a)–(c) from 7.2 are satisfied and endow  $G$  with the  $\mathbb{K}$ -analytic Lie group structure described there. Let  $\mathcal{O}$  be the topology on the Lie group  $G$ . Then product sets are large in  $(G, \mathcal{O}) = \bigcup_{n \in \mathbb{N}} G_n$ . As a consequence,  $\mathcal{O} = \mathcal{O}_{TG}$  holds, i.e.,  $\mathcal{O}$  makes  $G$  the direct limit topological group  $\lim_{\rightarrow} G_n$ . If, moreover, the direct sequence  $G_1 \subseteq G_2 \subseteq \cdots$  is strict, then  $(G, \overrightarrow{\mathcal{O}})$  is complete.*

Before we prove Proposition 7.3, let us compile useful facts concerning the Baker-Campbell-Hausdorff (BCH-) multiplication.

**7.4** (See [12]). Let  $\mathfrak{g}$  be a Banach-Lie algebra and  $\|\cdot\|$  be a norm on  $\mathfrak{g}$  which is *compatible* in the sense that it defines the topology of  $\mathfrak{g}$  and  $\|[x, y]\| \leq \|x\| \|y\|$  holds for all  $x, y \in \mathfrak{g}$ . Then the BCH-series converges for  $x, y \in \mathfrak{g}$  with  $\|x\| + \|y\| < \ln \frac{3}{2}$  and defines an analytic function

$$\{(x, y) \in \mathfrak{g} \times \mathfrak{g} : \|x\| + \|y\| < \ln(3/2)\} \rightarrow B_{\ln 2}^{\mathfrak{g}}(0), \quad (x, y) \mapsto x * y.$$

If  $\mathfrak{g} = L(G)$  for some Banach-Lie group  $G$ , then

$$\exp_G(x * y) = \exp_G(x) \exp_G(y) \quad \text{for all } x, y \in \mathfrak{g} \text{ with } \|x\| + \|y\| < \ln \frac{3}{2}. \quad (12)$$

See [16, Lemma 3.5 (a)] for the following estimates concerning derivatives of the BCH-multiplication:

**Lemma 7.5** *There exists  $s_0 \in ]0, \frac{1}{3} \ln \frac{3}{2}[$  such that, for each Banach-Lie algebra  $\mathfrak{g}$  and compatible norm  $\|\cdot\|$  on  $\mathfrak{g}$ ,*

$$(\forall x, y \in B_{s_0}^{\mathfrak{g}}(0)) \quad \|(\mu^{\mathfrak{g}})'(x, y) - \alpha^{\mathfrak{g}}\|_{op} \leq \frac{1}{2}, \quad (13)$$

where

$$\alpha^{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}, \quad (x, y) \mapsto x + y$$

is the addition map and

$$\mu^{\mathfrak{g}} : B_{s_0}^{\mathfrak{g}}(0) \times B_{s_0}^{\mathfrak{g}}(0) \rightarrow \mathfrak{g}, \quad (x, y) \mapsto x * y$$

the BCH-multiplication. □

To calculate the operator norm, the maximum norm was used on  $\mathfrak{g} \times \mathfrak{g}$  here.

With  $s_0$  and notation as in Lemma 7.5, we deduce:

**Lemma 7.6** *For each Banach-Lie algebra  $\mathfrak{g}$  and compatible norm  $\|\cdot\|$  on  $\mathfrak{g}$ , we have*

$$x * y + B_{r/2}^{\mathfrak{g}}(0) \subseteq x * B_r^{\mathfrak{g}}(y) \subseteq x * y + B_{3r/2}^{\mathfrak{g}}(0) \quad (14)$$

for all  $x \in B_{s_0}^{\mathfrak{g}}(0)$ ,  $y \in B_{s_0/2}^{\mathfrak{g}}(0)$  and  $r \in ]0, \frac{s_0}{2}]$ .

**Proof.** Setting  $R(x, y) := \mu^{\mathfrak{g}}(x, y) - x - y$ , we have

$$\mu^{\mathfrak{g}}(x, y) = x + y + R(x, y) \quad \text{for all } x, y \in B_{s_0}^{\mathfrak{g}}(0).$$

Since  $\|R'(x, y)\|_{op} \leq \frac{1}{2}$  for all  $(x, y) \in B_{s_0}^{\mathfrak{g}}(0) \times B_{s_0}^{\mathfrak{g}}(0)$  by (13) and the latter set is convex, Lemma A.7(b) shows that

$$\text{Lip}(R) \leq \frac{1}{2}.$$

For  $x \in B_{s_0}^{\mathfrak{g}}(0)$ , consider the map

$$\mu_x^{\mathfrak{g}}: B_{s_0}^{\mathfrak{g}}(0) \rightarrow \mathfrak{g}, \quad y \mapsto \mu^{\mathfrak{g}}(x, y).$$

For all  $y, z \in B_{s_0}^{\mathfrak{g}}(0)$ , we have

$$\begin{aligned} \|\mu_x^{\mathfrak{g}}(z) - \mu_x^{\mathfrak{g}}(y) - \text{id}_{\mathfrak{g}}(z - y)\| &= \|\mu^{\mathfrak{g}}(x, z) - (x + z) - \mu^{\mathfrak{g}}(x, y) + x + y\| \\ &= \|R(x, z) - R(x, y)\| \leq \text{Lip}(R)\|z - y\| \end{aligned}$$

and thus  $\text{Lip}(\mu_x^{\mathfrak{g}} - \text{id}_{\mathfrak{g}}) \leq \text{Lip}(R) \leq \frac{1}{2}$ . Applying now the Quantitative Inverse Function Theorem [25, Lemma 6.1 (a)] (or the version in [52]) to the function  $\mu_x^{\mathfrak{g}}$  with  $A := \text{id}_{\mathfrak{g}}$ , we get (14).  $\square$

**Proof of Proposition 7.3.** To see that product sets are large in  $G = \bigcup_{n \in \mathbb{N}} G_n$ , let  $(U_n)_{n \in \mathbb{N}}$  be a sequence of identity neighbourhoods  $U_n \subseteq G_n$ . By hypothesis,

$$\|x\|_m \leq \|x\|_k \quad \text{for all integers } 1 \leq k \leq m \text{ and all } x \in \mathfrak{g}_k. \quad (15)$$

Let  $s_0$  be as in Lemma 7.5. For  $n \in \mathbb{N}$ , choose

$$r_n \in ]0, s_0/2^{n+1}[ \quad (16)$$

so small that

$$V_n := \exp_{G_n}(B_{r_n}^{\mathfrak{g}_n}(0)) \subseteq U_n. \quad (17)$$

Write  $x *_n y := \mu^{\mathfrak{g}_n}(x, y)$  for the BCH-multiplication, for  $x, y \in B_{s_0}^{\mathfrak{g}_n}(0)$  (as in Lemma 7.5). Define

$$W_1 := B_{r_1}^{\mathfrak{g}_1}(0).$$

We claim that

$$W_n := W_{n-1} *_n B_{r_n}^{\mathfrak{g}_n}(0) \quad (18)$$

can be defined for each integer  $n \geq 2$ , and

$$\sum_{k=1}^n B_{r_k/2}^{\mathfrak{g}_k}(0) \subseteq W_n \subseteq \sum_{k=1}^n B_{3r_k/2}^{\mathfrak{g}_k}(0). \quad (19)$$

If the claim is true, then  $W := \bigcup_{n \in \mathbb{N}} W_n$  is a 0-neighbourhood in  $\mathfrak{g}$ , as it contains the convex set

$$S := \bigcup_{n \in \mathbb{N}} (B_{r_1/2}^{\mathfrak{g}_1}(0) + \cdots + B_{r_n/2}^{\mathfrak{g}_n}(0))$$

which is an open 0-neighbourhood in the locally convex direct limit  $\mathfrak{g} = \bigcup_{n \in \mathbb{N}} \mathfrak{g}_n$  as it intersects each  $\mathfrak{g}_n$  in an open 0-neighbourhood. Since  $\exp_G(W)$  contains the open subset  $\exp_G(S \cap Q)$  of  $G$  (with  $Q$  as in 7.2), we deduce that  $\exp_G(W)$  is an identity neighbourhood in  $G$ . Now

$$\exp_G(W_n) = V_1 V_2 \cdots V_n$$

for each  $n \in \mathbb{N}$ ; this is trivial if  $n = 1$  and follows inductively as

$$\begin{aligned} \exp_G(W_n) &= \exp_{G_n}(W_n) = \exp_{G_n}(W_{n-1} *_n B_{r_n}^{\mathfrak{g}_n}(0)) \\ &= \exp_{G_n}(W_{n-1}) \exp_{G_n}(B_{r_n}^{\mathfrak{g}_n}(0)) = \exp_{G_{n-1}}(W_{n-1}) V_n \\ &= V_1 \cdots V_{n-1} V_n, \end{aligned}$$

using (12), the definition of  $V_n$ , and the inductive hypothesis. Thus

$$U := \bigcup_{n \in \mathbb{N}} U_1 \cdots U_n \supseteq \bigcup_{n \in \mathbb{N}} V_1 \cdots V_n = \bigcup_{n \in \mathbb{N}} \exp_G(W_n) = \exp_G(W),$$

whence  $U$  is an identity neighbourhood in  $G$  and so product sets are large.

We now prove the claim, by induction. For  $n = 2$ , we can form

$$W_2 := W_1 *_2 B_{r_2}^{\mathfrak{g}_2}(0)$$

as  $W_1 = B_{r_1}^{\mathfrak{g}_1}(0) \subseteq B_{s_0}^{\mathfrak{g}_1}(0) \subseteq B_{s_0}^{\mathfrak{g}_2}(0)$  by (15), and  $B_{r_2}^{\mathfrak{g}_2}(0) \subseteq B_{s_0}^{\mathfrak{g}_2}(0)$ .

Moreover, as  $r_2 \leq s_0/2$ , we have

$$W_1 + B_{r_2/2}^{\mathfrak{g}_2}(0) \subseteq W_2 \subseteq W_1 + B_{3r_2/2}^{\mathfrak{g}_2}(0)$$

by (14). Hence

$$B_{r_1/2}^{\mathfrak{g}_1}(0) + B_{r_2/2}^{\mathfrak{g}_2}(0) \subseteq W_2 \subseteq B_{3r_1/2}^{\mathfrak{g}_1}(0) + B_{3r_2/2}^{\mathfrak{g}_2}(0)$$

a fortiori. For the induction step, assume that  $n \geq 2$  and that  $W_1, \dots, W_n$  have already been defined such that (19) holds with  $k \in \{1, \dots, n\}$  in place of  $n$ . In particular, (19) holds for  $n$  and its right hand side is a subset of

$$\sum_{k=1}^n B_{s_0/2^k}^{\mathfrak{g}_k}(0) \subseteq \sum_{k=1}^n B_{s_0/2^k}^{\mathfrak{g}_{n+1}}(0) \subseteq B_{s_0}^{\mathfrak{g}_{n+1}}(0),$$

using (15) for the first inclusion. Thus  $W_n \subseteq B_{s_0}^{\mathfrak{g}_{n+1}}(0)$  and since  $r_{n+1} \leq s_0$ , we deduce that  $W_{n+1} := W_n *_{n+1} B_{r_n}^{\mathfrak{g}_{n+1}}(0)$  can be defined. Moreover,

$$W_n + B_{r_{n+1}/2}^{\mathfrak{g}_{n+1}}(0) \subseteq W_{n+1} \subseteq W_n + B_{3r_{n+1}/2}^{\mathfrak{g}_{n+1}}(0),$$

by (14). Using (19), we deduce that

$$\sum_{k=1}^{n+1} B_{r_k/2}^{\mathfrak{g}_k}(0) \subseteq W_{n+1} \subseteq \sum_{k=1}^{n+1} B_{3r_k/2}^{\mathfrak{g}_k}(0),$$

which completes the inductive proof of the claim.

As product sets are large in  $G = \bigcup_{n \in \mathbb{N}} G_n$  by the preceding, the last and penultimate assertion of the proposition follow from [26, Proposition 11.8] and Theorem A, respectively.  $\square$

## A Infinite-dimensional calculus and Lie groups

We are using a setting of  $C^k$ -maps between open subsets of locally convex spaces which goes back to A. Bastiani [8] and is also known as Keller's  $C_c^k$ -theory. See [20], [30], [31], [39] and [40] for streamlined introductions (cf. also [9]).

**A.1** If  $E$  and  $F$  are Hausdorff locally convex spaces,  $U \subseteq E$  is open and  $k \in \mathbb{N}_0 \cup \{\infty\}$ , then a map  $f: U \rightarrow F$  is called  $C^k$  if it is continuous, the iterated directional derivatives

$$d^{(j)}f(x, y_1, \dots, y_j) := (D_{y_j} \cdots D_{y_1} f)(x)$$

exist for all  $j \in \mathbb{N}$  such that  $j \leq k$ , all points  $x \in U$  and all directions  $y_1, \dots, y_j \in E$ , and the maps  $d^{(j)}f: U \times E^j \rightarrow F$  so obtained are continuous. If  $k \geq 1$ , then a map  $f$  as before is  $C^k$  if and only if  $f$  is  $C^1$  and  $df :=$



$d^{(1)}f: U \times E \rightarrow F$  is  $C^{k-1}$  (see, e.g., [20] or [30]). If  $f: U \rightarrow F$  is a  $C^1$ -map and  $x \in U$ , we sometimes write  $f'(x)$  for the map

$$f'(x) := df(x, \bullet): E \rightarrow F, \quad y \mapsto df(x, y)$$

which is continuous linear (see, e.g., [20] or [30]).

If  $E$  and  $F$  are Banach spaces and  $f: U \rightarrow F$  is a mapping on an open subset  $U \subseteq E$ , then every  $C^{k+1}$ -map with  $k \in \mathbb{N}$  is  $k$  times continuously Fréchet differentiable in the classical sense (abbreviated  $FC^k$  now); conversely, every  $FC^k$ -map is  $C^k$  (see, e.g., [37] or [51]).

**A.2** Let  $E$  and  $F$  be Hausdorff locally convex spaces over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  and  $U \subseteq E$  be an open set. If  $\mathbb{K} = \mathbb{C}$ , then a function  $f: U \rightarrow F$  is called *complex analytic* if it is continuous and, for each  $x \in U$ , there exists an open neighbourhood  $Y \subseteq U$  of  $x$  and a sequence  $(p_n)_{n \in \mathbb{N}_0}$  of continuous complex homogeneous polynomials  $p_n: E \rightarrow F$  of degree  $n$  such that

$$(\forall y \in Y) \quad f(y) = \sum_{n=0}^{\infty} p_n(y - x)$$

as a pointwise limit. If  $\mathbb{K} = \mathbb{R}$ , then  $f$  is called *real analytic* if it extends to a complex analytic  $F_{\mathbb{C}}$ -valued mapping on some open neighbourhood of  $U$  in  $E_{\mathbb{C}}$  (see, e.g., [20] or [30]).

The identity theorem holds for  $\mathbb{K}$ -analytic maps. Every  $\mathbb{K}$ -analytic map is  $C^\infty$  in particular. If  $E$  and  $F$  are Banach spaces, then the preceding notion of  $\mathbb{K}$ -analyticity is equivalent to the one used in [12] (cf. [30] or [10]).

**A.3** Since compositions of composable  $C^k$ -maps (resp.,  $\mathbb{K}$ -analytic maps) are  $C^k$  (resp.,  $\mathbb{K}$ -analytic),  $C^k$ -manifolds (resp.,  $\mathbb{K}$ -analytic manifolds) modelled on a Hausdorff locally convex space  $E$  (and hence also smooth Lie groups and  $\mathbb{K}$ -analytic Lie groups) can be defined as in the finite-dimensional case, replacing  $\mathbb{R}^n$  with  $E$  in the definition of a manifold (see, e.g., [30], cf. also [9]).<sup>20</sup> In this article, we use the word “manifold” (resp., “Lie group”) for manifolds (resp. Lie groups) modelled on Hausdorff locally convex spaces.

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<sup>20</sup>Such manifolds are assumed Hausdorff, but neither second countable, paracompact, nor regular as topological spaces.

**A.4** For a  $C^k$ -manifold  $M$  modelled on a Hausdorff locally convex space with  $k \in \mathbb{N} \cup \{\infty\}$ , the tangent bundle  $TM$  can be defined (which is  $C^{k-1}$ ), and tangent maps  $Tf: TM \rightarrow TN$  for  $C^k$ -maps  $f: M \rightarrow N$  between  $C^k$ -manifolds; likewise for  $\mathbb{K}$ -analytic manifolds and  $\mathbb{K}$ -analytic maps (see, e.g., [30], cf. also [9]).

If  $G$  is a Lie group, we write  $L(G)$  (or also  $\mathfrak{g}$ ) for its tangent space at the neutral element  $e$ , endowed with the topological Lie algebra structure inherited from the Lie bracket of vector fields, via the identification of  $v \in T_e G$  with the corresponding left invariant vector field on  $G$ . As usual,  $L(f) := T_e f: L(G) \rightarrow L(H)$  if  $f: G \rightarrow H$  is a smooth group homomorphism between Lie groups. If a Lie group  $G$  admits a (necessarily unique) smooth group homomorphism  $\gamma_v: (\mathbb{R}, +) \rightarrow G$  with  $\gamma'_v(0) := L(\gamma_v)(1) = v$  for each  $v \in L(G)$ , then we say that  $G$  has an exponential function and define the latter as

$$\exp_G: L(G) \rightarrow G, \quad v \mapsto \gamma_v(1).$$

If  $G$  is a Banach-Lie group over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ , then  $G$  has an exponential function and  $\exp_G: \mathfrak{g} \rightarrow G$  is  $\mathbb{K}$ -analytic (see [12]).

**A.5** If  $E$  is a locally convex space and  $U \subseteq E$  an open subset, we identify the tangent bundle  $TU$  with  $U \times E$  in the usual way. If  $M$  is a  $C^1$ -manifold and  $f: M \rightarrow U$  a  $C^1$ -map, we write  $df: TM \rightarrow E$  for the second component of  $Tf: TM \rightarrow TU = U \times E$ . Thus

$$Tf = (f \circ \pi_{TM}, df)$$

with the bundle projection  $\pi_{TM}: TM \rightarrow M$ .

**A.6**  $C^k$ -maps  $f: U \rightarrow F$  can also be defined if  $U$  in A.1 is only a subset of  $E$  which has dense interior  $U^0$  and is locally convex in the sense that every point  $x \in U$  has a convex relatively open neighbourhood in  $U$ . In this case,  $f$  is called  $C^k$  if  $f$  is continuous,  $f|_{U^0}$  is  $C^k$  and the maps  $d^{(j)}(f|_{U^0}): U^0 \times E^j \rightarrow F$  have continuous extensions  $d^{(j)}f: U \times E^j \rightarrow F$  for all  $j \in \mathbb{N}$  such that  $j \leq k$ . In particular, this enables manifolds with boundary (modelled on Hausdorff locally convex spaces) to be defined (see [30]).

**Lemma A.7** *Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be normed spaces,  $U \subseteq E$  be an open subset and  $f: U \rightarrow F$  be a  $C^1$ -map. Then the following holds:*

- (a)  $f$  is locally Lipschitz.  
(b) If  $U$  is convex, then  $f$  is Lipschitz if and only if

$$L := \sup_{x \in U} \|f'(x)\|_{op} < \infty. \quad (20)$$

In this case,  $\text{Lip}(f) = L$ .

**Proof.** (b) For  $x \in U$  and  $y \in E$ , we have

$$\|t^{-1}(f(x + ty) - f(x))\|_F = |t|^{-1} \|f(x + ty) - f(x)\| \leq \text{Lip}(f) \|y\|_E$$

for small  $0 \neq t \in \mathbb{R}$  if  $f$  is Lipschitz. Passing to  $t \rightarrow 0$ , we find that

$$\|f'(x)(y)\|_F = \|df(x, y)\|_F \leq \text{Lip}(f) \|y\|_E,$$

whence  $\|f'(x)\|_{op} \leq \text{Lip}(f)$  and thus  $L \leq \text{Lip}(f)$ . Conversely, assume that the supremum  $L$  in (20) is finite. For all  $x, y \in U$ , using the Mean Value Theorem (see, e.g., [30, Proposition 1.2.6]) we get

$$\begin{aligned} \|f(y) - f(x)\|_F &= \left\| \int_0^1 df(x + t(y - x), y - x) dt \right\|_F \\ &\leq \int_0^1 \|f'(x + t(y - x))\|_{op} \|y - x\|_E dt \leq L \|y - x\|_E, \end{aligned}$$

whence  $f$  is Lipschitz with  $\text{Lip}(f) \leq L$ . The assertions follow.

(a) Given  $x \in U$ , by continuity of  $df: U \times E \rightarrow F$  there is an open convex neighbourhood  $Y \subseteq U$  of  $x$  and  $r > 0$  such that

$$df(Y \times \overline{B}_r^E(0)) \subseteq \overline{B}_1^F(0).$$

Hence  $\|f'(y)\|_{op} \leq \frac{1}{r}$  for all  $y \in Y$ , and thus  $f|_Y$  is Lipschitz (by (a)).  $\square$

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